

A note on a Cayley graph of \mathfrak{S}_n .

Guillaume Chapuy,
CNRS - LIAFA, Université Paris 7,
Paris, France.

Valentin Féray,
CNRS - LaBRI, Université Bordeaux 1,
Talence, France.

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Abstract

Recently in graph theory several authors have studied the spectrum of the Cayley graph of the symmetric group \mathfrak{S}_n generated by the transpositions $(1, i)$ for $2 \leq i \leq n$. Several conjectures were made and partial results were obtained.

The purpose of this note is to point out that, as mentioned also by P. Renteln, this problem is actually already solved in another context. Indeed it is equivalent to studying the spectrum of so-called Jucys-Murphy elements in the algebra of the symmetric group, which is well understood. The aforementioned conjectures are direct consequences of the existing theory. We also present a related result from P. Biane, giving an asymptotic description of this spectrum.

We insist on the fact that this note does not contain any new results, but has only been written to convey the information from the algebraic combinatorics community to graph theorists.

1 Jucys-Murphy elements.

We let $n \geq 1$ be an integer and \mathfrak{S}_n be the symmetric group on $\{1, 2, \dots, n\}$. We let $\mathcal{P}(n)$ be the set of partitions of n and we note $\lambda \vdash n$ if $\lambda \in \mathcal{P}(n)$. The irreducible representations of \mathfrak{S}_n are canonically indexed by elements of $\mathcal{P}(n)$, and we denote by V_λ the irreducible module associated with the partition λ . The regular representation of \mathfrak{S}_n is decomposed into irreducible submodules as follows [14, Proposition 1.10.1]:

$$\mathbb{C}[\mathfrak{S}_n] = \bigoplus_{\lambda \in \mathcal{P}(n)} V_\lambda^{f_\lambda}, \quad (1)$$

where $f_\lambda := \dim V_\lambda$.

A partition $\lambda \vdash n$ is represented by its Ferrers diagram as on Figure 1(a). A *standard Young tableau* (SYT) of shape λ is a filling of the Ferrers diagram of λ with the elements $\{1, 2, \dots, n\}$ in such a way that elements increase along rows and columns (in particular all elements are distinct, and each element appears exactly once). See Figure 1(b). We let $\mathcal{T}(\lambda)$ be the set of SYT of shape λ . A well known result asserts [14, Theorem 2.6.5] that $f_\lambda = \text{card } \mathcal{T}(\lambda)$.

Let λ be a partition, and let \square be a box of the Ferrers diagram of T . Let $x(\square)$ and $y(\square)$ be its abscissa and ordinate, respectively, and define the *content* of the box as $c(\square) = y(\square) - x(\square)$. If $T \in \mathcal{T}(\lambda)$ is a SYT, and if i is an element of $\{1, 2, \dots, n\}$, we define $c_T(i)$ as the content of the box in which the label i appears in the tableau T . For example on Figure 1(b) one has $c_T(2) = 1$, $c_T(5) = 0$ and $c_T(6) = -2$.

The *Jucys-Murphy elements* J_2, \dots, J_n are elements of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ introduced separately by A. Jucys [8] and G. Murphy [11]. In recent years they have proved to be very

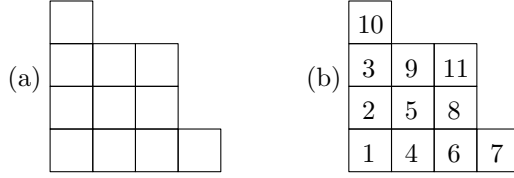


Figure 1: (a) The Ferrers diagram of the partition $\lambda = (4, 3, 3, 1)$ (we use the French notation); (b) A standard Young Tableau.

efficient tools in the study of the representations of the symmetric group [12] (see also [4]). They are defined by:

$$J_i = (1, i) + (2, i) + \cdots + (i - 1, i).$$

As shown by the following theorem [9, equation (12)], the action of the J_i 's on the irreducible modules V_λ is diagonal, and the eigenvalues have a combinatorial description in terms of contents (see also [5, Theorem 3.7], where the Cayley graph of S_n is explicitly mentioned):

Theorem 1.1 ([9]) *Let $\lambda \in \mathcal{P}(n)$. Then there exists a basis $(v_T)_{T \in \mathcal{T}(\lambda)}$ of the irreducible module V_λ , indexed by SYT's of shape λ , such that for all $i \in \{2, \dots, n\}$, one has:*

$$J_i v_T = c_T(i) v_T.$$

2 A Cayley Graph

As in [7, 1, 10], we consider the Cayley graph G_n on \mathfrak{S}_n generated by the transpositions $(1, i)$ for $2 \leq i \leq n$. The *spectrum* of this graph is defined as the spectrum of its adjacency matrix. It (evidently) coincides with the spectrum of the left multiplication by the Jucys-Murphy element J_n on the group algebra $\mathbb{C}[\mathfrak{S}_n]$ (this link and the fact that it implies that the aforementioned spectrum is integral are written explicitly in the introduction of paper [13]). From (1) and Theorem 1.1 we immediately obtain:

Corollary 2.1 *The spectrum of G_n contains only integers. The multiplicity $\text{mul}(k)$ of an integer $k \in \mathbb{Z}$ is given by:*

$$\text{mul}(k) = \sum_{\lambda \in \mathcal{P}(n)} f_\lambda I_\lambda(k)$$

where $I_\lambda(k) = \text{card}\{T \in \mathcal{T}(\lambda), c_T(n) = k\}$ is the number of standard Young tableaux of shape λ in which the integer n appears in a box of content k , and f_λ is the number of SYT of shape λ .

We list some direct remarks and consequences:

1. All eigenvalues are integers, as conjectured in [1, Conjecture 2.14].
2. The spectrum of G_n is contained in $\{-(n-1), \dots, n-1\}$. Moreover one has $\text{mul}(-n+1) = \text{mul}(n-1) = 1$.
3. Unless $n = 2$ or $n = 3$, one has $\text{mul}(0) \neq 0$ (as proved in [10]). Indeed unless $n = 2$ or $n = 3$, there exists a SYT of size n in which n appears on the main diagonal. More generally, the spectrum is given by:

$$\begin{aligned} \{k, \text{mul}(k) \neq 0\} &= \{-(n-1), \dots, n-1\} \setminus \{0\} \text{ if } n \in \{2, 3\}, \\ &\quad \{-(n-1), \dots, n-1\} \text{ if } n > 3. \end{aligned}$$

4. Let $1 \leq l \leq n$, and consider the “hook-shaped” partition $\lambda^{(l)} = (n-l, 1, 1, \dots, 1)$ of n . The dimension of this partition is $f_{\lambda^{(l)}} = \binom{n-1}{l-1}$, as can be seen directly or via the hook-length formula ([6] or [14, Theorem 3.10.2]). Moreover, the number of standard Young tableaux T of shape $\lambda^{(l)}$ such that $c_T(n) = l$ (i.e., such that n appears in the topmost box) equals $\binom{n-2}{l-1}$, since such a tableau is determined by the choice of the $l-1$ elements appearing between 1 and n on the left row of the tableau. Hence by Corollary 2.1 one has:

$$\text{mul}(l) \geq \binom{n-2}{l-1} \binom{n-1}{l}.$$

This improves the bound $\text{mul}(l) \geq \binom{n-2}{l-1}$ proved in [10]. Similar arguments (transpose tableaux) show that the same bound holds for $\text{mul}(-l)$.

5. The bound given in point 4 could possibly be refined by taking more complicated tableaux than hook-shapes into account. Rather than pursuing in this direction, we will show in the next section that almost all the eigenvalues are of order $O(\sqrt{n})$, and we will give a very precise description of the spectrum of G_n in this range of values.

3 Semi-circle law

To be comprehensive on what is known about the spectrum of J_n , we present here an asymptotic result of P. Biane [2, Theorem 1]: the spectral measure converges in distribution to the semicircle law. This provides a good description of the spectrum in the range $k = \Theta(\sqrt{n})$.

The proof technique is standard and goes back to the beginning of random matrix theory. As it is quite short and elegant, we repeat it here.

For $x \in \mathbb{R}$ we denote by δ_x the Dirac measure at x .

Proposition 3.1 *The measure*

$$\text{sp}_n := \frac{1}{n!} \sum_{k \in \mathbb{Z}} \text{mul}(k) \delta_{\frac{k}{2\sqrt{n}}}$$

converges in distribution to the semi-circle law, i.e. for all $a < b$ we have the convergence:

$$\frac{1}{n!} \sum_{k \in [2a\sqrt{n}, 2b\sqrt{n}] \cap \mathbb{Z}} \text{mul}(k) \longrightarrow \frac{2}{\pi} \int_a^b \sqrt{1 - \alpha^2} d\alpha$$

when n tends to infinity.

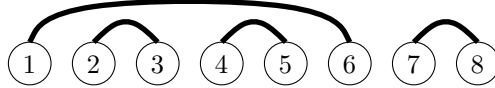
Proof. [sketch] It is enough to prove (see, e.g., [3, Theorem 30.2]) the convergence of moments, i.e. to prove that for any fixed $k \in \mathbb{Z}$ one has when n tends to infinity:

$$\frac{1}{n!} \sum_{l \in \mathbb{Z}} \text{mul}(l) \left(\frac{l}{2\sqrt{n}} \right)^k = \frac{n^{-k/2}}{2^k n!} \text{Tr} J_n^k \longrightarrow \frac{2}{\pi} \int_{\mathbb{R}} \alpha^k \sqrt{1 - \alpha^2} d\alpha = \begin{cases} 0, & k \text{ odd} \\ \frac{1}{2^k} \text{Cat}(p) & k = 2p. \end{cases} \quad (2)$$

where $\text{Cat}(p) := \frac{(2p)!}{(p+1)!p!}$ is the p -th Catalan number. Let $\sigma \in \mathbb{S}_n$. The multiplication by σ acts by permutation on the canonical basis of the group algebra $\mathbb{C}[\mathbb{S}_n]$. Therefore $\text{Tr} \sigma$ equals the number of fixed points under this action, i.e.:

$$\text{Tr} \sigma = \text{card}\{\mu \in \mathbb{S}_n, \sigma\mu = \mu\} = \begin{cases} n!, & \sigma = id \\ 0 & \sigma \neq id. \end{cases}$$

By developing the product $J_n^k = ((1\ n) + \dots + (n-1\ n))^k$, this implies that $\frac{1}{n!} \text{Tr} J_n^k$ equals the number of k -uples $(i_1, \dots, i_k) \in \llbracket 1, n-1 \rrbracket^k$ such that $(i_1\ n)(i_2\ n) \dots (i_k\ n) = id$. This number is 0 if k is odd (signature), so we assume that $k = 2p$. Since k is fixed and we are interested in large n asymptotics, we only need to consider the cases where the set of values $I := \{i_l, l = 1, 2, \dots, k\}$ has the largest cardinality. It is easily seen that each transposition $(i\ n)$ must appear an even number of times in the product, so the maximal case is $\text{card } I = p$. In this case, we consider the pairing of elements of $\{1, 2, \dots, k\}$ defined by pairing l and m if and only if $i_l = i_m$. It is convenient to represent this pairing by a diagram. For example, here is a drawing of this diagram in the case when $k = 8$ and the product has the form $(i_1\ n)(i_2\ n)(i_2\ n)(i_4\ n)(i_4\ n)(i_1\ n)(i_7\ n)(i_7\ n) = id$:



In general it is easily seen that this pairing is necessarily non-crossing. Conversely, there are exactly $(n-1)(n-2) \dots (n-p)$ ways to reconstruct a factorisation $(i_1\ n)(i_2\ n) \dots (i_k\ n) = id$ from one of the $Cat(p)$ non-crossing pairings of the set $\{1, 2, \dots, 2p\}$. Therefore one has:

$$\begin{aligned} \frac{1}{n!} \text{Tr} J_n^k &= (n-1)(n-2) \dots (n-p) Cat(p) + O(n^{p-1}) \\ &\sim n^p Cat(p), \end{aligned}$$

and the convergence of moments (2) is proved. □

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